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A Stochastic Differential Equation Model with Jumps for Fractional Advection and Dispersion

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Abstract The path of a tracer particle through a porous medium is typically modeled by a stochastic differential equation (SDE) driven by Brownian noise. This model may not be adequate for highly heterogeneous media. This paper extends the model to a general SDE driven by a Lévy noise. Particle paths follow a Markov process with long jumps. Their transition probability density solves a forward equation derived here via pseudo-differential operator theory and Fourier analysis. In particular, the SDE with stable driving noise has a space-fractional advection-dispersion equation (fADE) with variable coefficients as the forward equation. This result provides a stochastic solution to anomalous diffusion models, and a solid mathematical foundation for particle tracking codes already in use for fractional advection equations.

Keywords Stochastic differential equation · Pure jump Lévy process · Infinitesimal generator · Forward equation · Jump Lévy diffusion

1 Introduction

A mechanical description of the path of a tracer through a porous medium requires identifying the relevant forces involved and then solving a classical differential equation for the individual trajectories of all tracer particles collectively. Since in a macroscopic flow, the number of such molecules is enormous, the method can be a complicated one. An alternative approach is the statistical mechanics paradigm, where the trajectories of the particles are considered to be a stochastic process. This idea has been discussed by Bhattacharya, Gupta and Sposito [5, 6, 11]. They provided a probabilistic model that enables one to avoid detailed mechanical description. In their molecular-stochastic approach the position vector of a particle is described as a Markov process with continuous sample path. With the velocity vector coding the drift of a particle and the diffusivity tensor representing both molecular

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diffusion and mechanical dispersion, the Markov process can be shown to be a unique solution of Itô's stochastic differential equation (SDE) driven by a Brownian motion. Further, the conditional probability density function of the Markov process solves an Eulerian forward equation, the advection dispersion equation (ADE) that forms the basis for most current hydrological models of contaminant transport [3, 10].

Recently, it has been observed that this advection dispersion equation may not be adequate for modeling heavy tailed contaminant transport in a heterogeneous porous media. For such flows a model has been advocated that describes the dispersive flux by a fractional space derivative (the space-fractional advection-dispersion equation or fADE [4]). In this case a random walk particle tracking approach, similar to the stochastic-molecular approach mentioned before, has been proposed where individual particle paths follow a Markov process (Zhang et al. [21–23]). An SDE driven by a stable-Lévy process is the primary tool needed for the particle tracking method of solving the fADE [21]. In this article we lay out the mathematical foundation of this approach.

We start by deriving the governing forward equation of an SDE driven by a general jump Lévy process. The general theory can be applied to explain various anomalous diffusion models including tempered anomalous diffusion models [8, 16], another jump Lévy process that has useful application in geophysics. Next, as a special case, we derive the governing forward equation of an SDE driven by a stable-Lévy process that can be used for heavy tailed particle tracking.

To simplify the presentation we restrict attention to one dimension. The theory can be easily extended to time homogeneous SDE in $\mathbb{R}_d$.

The SDE driven by a general jump Lévy process and the associated governing equations are discussed in Sect. 2. In Sect. 3 we discuss an SDE driven by an α-Stable Lévy process as a special application of the concepts presented in Sect. 2. In this case the associated forward equation leads to the space-fractional advection-dispersion (fADE) equation used for modeling heavy tailed flows.

2 Stochastic Diffusion Driven by a Pure Jump Lévy Processes and Associated Governing Equations

The main objective of this section is to show that the transition probability density function of a diffusion process driven by a jump Lévy process satisfies a deterministic differential equation (forward equation) analogous to the governing equations used in various Markovian anomalous diffusion models [8, 16].

Let $\{X_t\}$ be a pure jump Lévy process with jump intensity (Lévy measure) ν (see Sato [19], p. 38). A diffusion equation, driven by the pure jump Lévy process X_t, is an SDE of the form:

$$dY_t = a(Y_t)dt + b(Y_t)dX_t \tag{2.1}$$

where, the coefficient function $a : \mathbb{R} \to \mathbb{R}$ is the drift coefficient that governs the average velocity of the plume and the coefficient function $b : \mathbb{R} \to \mathbb{R}$ is the diffusion coefficient that describes the spread of the plume. If the drift and diffusion coefficients satisfy a growth condition (as in (A.1)) and are Lipschitz continuous (as in (A.2)), then it can be shown (see Theorem A.1) that there exists a unique time homogeneous Markov process that solves the diffusion equation (2.1). This Markov process is explicitly constructed in Appendix A. Also the solution is continuous with respect to the initial starting value.

Let $P(t, y, x)$ be the transition probability of the Markov process $\{Y_t\}$, i.e, for a measurable set B of $\mathbb{R}$, we have $P(t, y, B) = P(Y_t \in B | Y_0 = y)$. More precisely, in case Y_t has a transition density $p_y(x, t)$, given $Y_0 = y$, we have $P(t, y, B) = \int_B P(t, y, dx) = \int_B p_y(t, x)dx$.

For any continuous functions f on $\mathbb{R}$, let $E^y[f(Y_t)] = \int_{\mathbb{R}} f(x)P(t, y, dx)$ be the expectation of $f(Y_t)$, under the condition $Y_0 = y$. The infinitesimal generator of the Markov process $\{Y_t\}$ is defined as:

$$Af = \lim_{t \downarrow 0} \frac{E^y f(Y_t) - f(y)}{t}.$$

Its domain $\mathcal{D}_A$ consists of all continuous function f for which the above limit exists. The infinitesimal generator can be derived by using the Itô formula for Lévy type stochastic integrals [13].

Theorem 2.1 (Infinitesimal Generator) *Let* $\{Y_t\}$ *be the solution process of the SDE* (2.1). *Also, suppose that the coefficient functions a and b satisfy conditions* (A.1) *and* (A.2). *If* $C_\nu = \int_{|x|\geq 1} x\nu(dx) < \infty$, *then for any twice differentiable function* f *with compact support, the infinitesimal generator A of* $\{Y_t\}$ *is given by*:

$$\begin{aligned} Af(y) &= f'(y)\big(a(y) + C_\nu b(y)\big) \\ &\quad + \int_{\mathbb{R}_0} \big\{f\big(y + b(y)x\big) - f(y) - f'(y)b(y)x\big\}\nu(dx) \end{aligned} \tag{2.2a}$$

alternatively, if $K_\nu = \int_{0<|x|<1} x\nu(dx) < \infty$, *then the infinitesimal generator is given by*:

$$Af(y) = f'(y)\big(a(y) - K_\nu b(y)\big) + \int_{\mathbb{R}_0} \big\{f\big(y + b(y)x\big) - f(y)\big\}\nu(dx). \tag{2.2b}$$

For the infinitesimal generator of any Markov process, in particular for the solution process of the SDE driven by a pure jump Lévy process, the following backward equation holds. The backward equation governs the evolution of transition probability $P(t, y, x)$ backward in time, since the variable y gives the initial location.

Theorem 2.2 (The backward equation) *Let A be the infinitesimal generator as in Theorem* 2.1 *for the solution process* $\{Y_t\}$ *of the SDE* (2.1). *Let* f *be a real valued twice differentiable function with compact support. For the transition probability* $P(t, y, x)$ *of* Y_t *given* $Y_0 = y$, *let us define*, $u(y,t) = \int_{\mathbb{R}} f(x)P(t, y, dx)$. *Then*, $\frac{\partial u}{\partial t}$ *exists and*

$$\frac{\partial u}{\partial t} = A(u) \tag{2.3}$$

where, *the R.H.S is to be interpreted as A applied to the function* $y \mapsto u(y, t)$.

Using Fourier analysis we can show that the infinitesimal generator in (2.2) is a pseudo differential operator (in sense of Jacob [14], Definition D.4), defined on the anisotropic Sobolev space $H^{\xi^2,2}(\mathbb{R})$ (definition in Appendix D.1).

Next, we show that the transition probability density function of the solution process of SDE (2.1) satisfies a deterministic differential equation *viz.* the *forward equation*. This forward equation is analogous to the Eulerian PDE that governs the path of the tracer particles in anomalous diffusion models. The forward equation can be derived from the backward equation using the infinitesimal generator in its pseudo differential operator form. The detailed derivation of the forward equation, along with the relevant Fourier analysis concepts, is included in the appendix (Sect. D.1).

Let us assume that there exists a transition probability density for the solution process $\{Y_t\}$ of the SDE in (2.1). Let $p_x(t, y)$ be the density of $y = Y_t$, given $Y_0 = x$. The forward equation governs the evolution of the transition probability density $p_x(t, y)$ forward in time, since the variable y here represents the location at time $t > 0$. To derive the forward equation we assume $p_x(t, y)$ belongs to the anisotropic Sobolev space $H^{\xi^2,2}(\mathbb{R})$.
Let us make a change of variable $b(y)x = -v$ in (2.2). Let $J(v)$ be the Jacobian of the transformation and define $\nu_1(y, dv) = J(v)\nu(\frac{-dv}{b(y)})$. We consider the case when the measure ν_1 is of the form $\nu_1(x, dy) = h(x)\mu(dy)$, where, h is a measurable function on $\mathbb{R}$ and μ is a measure on $\mathbb{R}$.

Theorem 2.3 (The forward equation) *For the coefficient functions a and b in the SDE* (2.1) *define* $M(x) := \max\{|a(x)|, |b(x)|^2\}$. *Let us assume the coefficient functions are such that*

$$\int_{\mathbb{R}} M^2(x)dx < \infty. \tag{2.4}$$

Let the Lévy measure ν *of the driving Lévy process satisfy* $\int_{|y|>1} \nu(dy) < \infty$. *Let* C_ν *and* K_ν *be as in Theorem* 2.1. *If* $C_\nu < \infty$, *then the transition probability density function of the solution process of SDE* (2.1) *satisfies the following equation*:

$$\begin{aligned}\frac{\partial}{\partial s} p_x(s, y) = &\int_{\mathbb{R}_0} \big[(p_x \cdot h)(s, y - r) - (p_x \cdot h)(s, y) + r(p_x \cdot h)'(s, y)\big]\mu(d(-r)) \\ &- \frac{\partial}{\partial y}(p_x \cdot G)(s, y)\end{aligned}$$

where, $(p_x \cdot h)(s, y) = h(y)p_x(s, y)$ *and* $G(y) = a(y) + C_\nu b(y)$. *Alternatively, if* $K_\nu < \infty$, *then*

$$\frac{\partial}{\partial s} p_x(s, y) = \int_{\mathbb{R}_0} \big[(p_x \cdot h)(s, y - r) - (p_x \cdot h)(s, y)\big]\mu(d(-r)) - \frac{\partial}{\partial y}(p_x \cdot H)(s, y)$$

where, $(p_x \cdot h)(s, y) = h(y)p_x(s, y)$ *and* $H(y) = a(y) - K_\nu b(y)$.

3 Application to Special Case: Diffusion Driven by α-Stable Lévy Processes

The fractional advection dispersion equation or fADE is used to model a variety of anomalous diffusion processes, where observations show that the plume spreads away from its center of mass faster than $(t^{1/2})$ scaling implied by the Brownian motion model. This is called a super diffusion. A diffusion equation driven by an α-Stable Lévy noise can be applied to these situations. We follow the stable distribution parameterizations in Samorodnitsky and Taqqu [17] and write $X \sim S_\alpha(\sigma, \beta, \mu)$ to denote that the random variable X follows a stable distribution (Definition E.1) with index of stability α, scale parameter σ, skewness parameter β and shift parameter μ. Where, $0 < \alpha \leq 2$, $\sigma \geq 0$, $-1 \leq \beta \leq 1$ and $\mu \in \mathbb{R}$. An α-Stable Lévy Process is a Lévy process $X_t^{(\alpha)}$, such that, $X_t^{(\alpha)} - X_s^{(\alpha)} \sim S_\alpha((t - s)^{1/\alpha}, \beta, 0)$;

for $0 \le s < t < \infty$ and $-1 \le \beta \le 1$. Note that a stable Lévy process is a jump Lévy process. Let us define,

$$I_\beta(u) = \begin{cases} (1+\beta), & \text{if } u > 0, \\ (1-\beta), & \text{if } u < 0 \end{cases}$$

and constant C_α as:

$$(C_\alpha)^\alpha = \begin{cases} (2\alpha^{-1}(\Gamma(1-\alpha))\cos\frac{\pi\alpha}{2})^{-1}, & \text{if } 0 < \alpha < 1, \\ (2\frac{\Gamma(2-\alpha)}{\alpha(\alpha-1)}(-\cos\frac{\pi\alpha}{2}))^{-1}, & \text{if } 1 < \alpha < 2. \end{cases}$$

Then it can be shown (Theorem E.1) that an α-stable Lévy process with skewness parameter β is a jump Lévy process with jump intensity (Lévy measure) ν_α, where $\nu_\alpha(dy) = I_\beta(y)(C_\alpha)^\alpha \frac{dy}{|y|^{1+\alpha}}$. If the drift coefficient a and the diffusion coefficient b satisfy the growth condition and Lipschitz condition as in Theorem A.1, then there exists a unique time homogeneous Markov process $\{Y_t\}$ that solves the SDE of form

$$dY_t = a(Y_t)dt + b(Y_t)dX_t^{(\alpha)}. \tag{3.1}$$

Since $\{Y_t\}$ is a Markov process, we can get corresponding infinitesimal generator and the associated forward and backward equations. For (3.1), the infinitesimal generator of the solution process can be expressed in terms of the fractional derivatives of order α. The forward equation in terms of fractional derivatives is used as the Langevin equation of heavy tailed ground water flows. A fractional derivative of order α can be defined as follows (see, e.g., Samko et al. [18]):

Definition 3.1 Let $\hat{f}$ be the Fourier transform of f (definition in Appendix D.1). Let us write $g(x) = \frac{\partial^\alpha}{\partial(\pm x)^\alpha} f(x)$, then $\hat{g}(\xi) = (\pm i\xi)^\alpha \hat{f}(\xi)$. The fractional derivative of order α for a function f is the inverse Fourier transform $g(x) := F^{-1}[(\pm i\xi)^\alpha \hat{f}(\xi)]$.

The next theorem gives the infinitesimal generator associated with the solution of an SDE driven by an α-Stable Lévy process. Detailed derivation is included in the appendix. In following theorems we shall use the notation $b^\alpha(\cdot)$ to express the following:

$$b^\alpha(\cdot) = \begin{cases} |b(\cdot)|^\alpha, & \text{if } b(\cdot) > 0, \\ -|b(\cdot)|^\alpha, & \text{if } b(\cdot) < 0. \end{cases}$$

Theorem 3.1 *Consider an SDE driven by an α-Stable Lévy process as in* (3.1). *Under the assumption of Theorem* A.1, *a unique Markov process $\{Y_t\}$ exists that solves* (3.1). *Then for a twice differentiable function f, the infinitesimal generator of $\{Y_t\}$ is given as follows*: *if* $0 < \alpha < 1$,

$$\begin{aligned} Af(y) = a(y)f'(y) + \Bigg[(1-\beta)\left(2\cos\left(\frac{\pi\alpha}{2}\right)\right)^{-1} b^\alpha(y)\frac{d^\alpha}{dy^\alpha} f(y) \\ + (1+\beta)\left(2\cos\left(\frac{\pi\alpha}{2}\right)\right)^{-1} b^\alpha(y)\frac{d^\alpha}{d(-y)^\alpha} f(y)\Bigg]. \end{aligned} \tag{3.2a}$$

if $1 < \alpha < 2$,

$$Af(y) = a(y)f'(y) + \left[(1-\beta)\left(-2\cos\left(\frac{\pi\alpha}{2}\right)\right)^{-1} b^{\alpha}(y)\frac{d^{\alpha}}{dy^{\alpha}}f(y) + (1+\beta)\left(-2\cos\left(\frac{\pi\alpha}{2}\right)\right)^{-1} b^{\alpha}(y)\frac{d^{\alpha}}{d(-y)^{\alpha}}f(y)\right]. \tag{3.2b}$$

The backward equation can be obtained using Theorem 2.2. Let A be the infinitesimal generator for the solution process $\{Y_t\}$ of an SDE driven by an α-Stable Lévy process as in (3.1) and $P(t, y, x)$ be the transition probability of Y_t. Let f be a twice differentiable real function with compact support.

Define, $u(y,t) = \int f(x)P(t, y, dx)$. Then, the backward equation is given as follows: if $0 < \alpha < 1$,

$$\frac{\partial u(y,t)}{\partial t} = a(y)u'(y,t) + \left[(1-\beta)\left(2\cos\left(\frac{\pi\alpha}{2}\right)\right)^{-1} b^{\alpha}(y)\frac{d^{\alpha}}{dy^{\alpha}}u(y,t) + (1+\beta)\left(2\cos\left(\frac{\pi\alpha}{2}\right)\right)^{-1} b^{\alpha}(y)\frac{d^{\alpha}}{d(-y)^{\alpha}}u(y,t)\right] \tag{3.3a}$$

if $1 < \alpha < 2$,

$$\frac{\partial u(y,t)}{\partial t} = a(y)u'(y,t) + \left[(1-\beta)\left(-2\cos\left(\frac{\pi\alpha}{2}\right)\right)^{-1} b^{\alpha}(y)\frac{d^{\alpha}}{dy^{\alpha}}u(y,t) + (1+\beta)\left(-2\cos\left(\frac{\pi\alpha}{2}\right)\right)^{-1} b^{\alpha}(y)\frac{d^{\alpha}}{d(-y)^{\alpha}}u(y,t)\right]. \tag{3.3b}$$

Theorem 2.3 can be used to obtain the forward equation. Let $\{Y_t\}$ be the solution process for the SDE in (3.1) . Let us further assume that there exists a transition probability density for $\{Y_t\}$ and $p_x(t, y)$ be the transition p.d.f of $y = Y_t$, given $Y_0 = x$. We have already stated that the Lévy measure for the α-Stable Lévy process $\{X_t\}$ is given by $\nu_\alpha(dy) = I_\beta(y)(C_\alpha)^\alpha \frac{dy}{|y|^{1+\alpha}}$.

Note that, the measure $\nu_1(x, du)$ in Theorem 2.3 is the measure derived from the Lévy measure $\nu_\alpha(dy)$ by change of variable $-u = b(x)y$. So in this case the change of variable leads to

$$\nu_1(x, du) = -I_\beta(-u)(C_\alpha)^\alpha b^\alpha(x)\frac{du}{|u|^{1+\alpha}}.$$

Now we see ν_1 is of the form $\nu_1(x, dy) = h(x)\mu(dy)$, where $h(x) = b^\alpha(x)$ and $\mu \equiv \nu_\alpha$. Further, $\int_{|y|>1} \nu_\alpha(dy) = (C_\alpha)^\alpha \int_{|y|\geq 1} I_\beta(y)\frac{dy}{|y|^{1+\alpha}} = 2\frac{(C_\alpha)^\alpha}{\alpha} < \infty$.

It can be easily verified that if $0 < \alpha < 1$, then $K_{\nu_\alpha} < \infty$, on the other hand, if $1 < \alpha < 2$, then $C_{\nu_\alpha} < \infty$ (use the definitions of K_ν, C_ν from Theorem 2.3). This demonstrates the assumptions made for Theorem 2.3 hold in this case.

Thus, if condition (2.4) holds, then the transition probability density function of Y_t satisfies the forward equation given in Theorem 2.3. This leads us to the next theorem.

Theorem 3.2 (The forward equation for an α-stable Lévy diffusion) *Consider an SDE driven by an α-stable Lévy process* ($\alpha \neq 1$, $0 < \alpha < 2$) *as in* (3.1). *Under the assumptions of Theorem* A.1, *the solution process* $\{Y_t\}$ *exists and is unique. If the coefficient functions a and b satisfy assumption* (2.4), *and if there exists a transition probability density function $p_x(s, y)$ of the solution process $\{Y_s\}$ given $Y_0 = x$, then the following forward equation* *holds*: *if* $0 < \alpha < 1$,

$$\frac{\partial u(y,t)}{\partial t} = a(y)u'(y,t) + \Bigg[(1-\beta)\left(2\cos\left(\frac{\pi\alpha}{2}\right)\right)^{-1} b^{\alpha}(y)\frac{d^{\alpha}}{dy^{\alpha}}u(y,t) + (1+\beta)\left(2\cos\left(\frac{\pi\alpha}{2}\right)\right)^{-1} b^{\alpha}(y)\frac{d^{\alpha}}{d(-y)^{\alpha}}u(y,t)\Bigg] \tag{3.4a}$$

if $1 < \alpha < 2$,

$$\frac{\partial u(y,t)}{\partial t} = a(y)u'(y,t) + \Bigg[(1-\beta)\left(-2\cos\left(\frac{\pi\alpha}{2}\right)\right)^{-1} b^{\alpha}(y)\frac{d^{\alpha}}{dy^{\alpha}}u(y,t) + (1+\beta)\left(-2\cos\left(\frac{\pi\alpha}{2}\right)\right)^{-1} b^{\alpha}(y)\frac{d^{\alpha}}{d(-y)^{\alpha}}u(y,t)\Bigg]. \tag{3.4b}$$

The forward equation in this form is the same as the fractional advection dispersion equation (fADE) used in particle tracking of ground water contamination modeling [23].

4 Concluding Remarks

In this article we construct a diffusion process driven by a jump Lévy noise such that its transition probability density function solves the governing Eulerian PDE in the anomalous diffusion models. The methods used in this paper are constructive and explicitly identify the solution of the governing equation based on the compound Poisson processes. This can be useful for simulation purposes. The theory presented here can be applied to flow equations other than the ones demonstrated. For example, a SDE with three terms including a simple drift, gives FD-ADE flow equation [21]. This flow can be associated with a Markov process and can be treated as a minor extension of the cases discussed in this paper. Our future work will demonstrate that the conditions applied to the drift and dispersion coefficients of the main SDE can be made less restrictive. That will enhance the utility of the model for the practical applications.

See [9] for method of simulating a solution process of an SDE driven by an α-Stable Lévy process in context of fADE model fitting to real data. Bottcher and Schilling [7] shows another concrete way to simulate a general SDE solution process driven by an arbitrary jump Lévy process.

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Appendix A

A.1 Existence, Uniqueness and Markov Property of jump Lévy Diffusion

Theorem A.1 *Let coefficient functions a and b of diffusion equation* (2.1) *satisfy the following growth condition and Lipschitz condition*

(A) *Growth condition*: $\forall y$ *in* $\mathbb{R}$, $\exists$ *constant* $C > 0$, *such that*,

$$|a(y)|^2 + |b(y)|^2 \leq C\left(1 + |y|^2\right). \tag{A.1}$$

(B) *Lipschitz condition*: $\forall y_1\ y_2$ *in* $\mathbb{R}$, $\exists$ *constant* $C' > 0$, *such that*,

$$|a(y_1) - a(y_2)|^2 + |b(y_1) - b(y_2)|^2 \leq C'\left(|y_1 - y_2|^2\right). \tag{A.2}$$

Then, there exists a unique solution process $\{Y_t\}$ *for the equation* (2.1); *also the solution process is continuous w.r.t the initial value. Further it can be shown that the solution is a time homogeneous Markov process.*

Proof of Theorem A.1 Let $\{X_t\}$ be a pure jump Lévy process with Lévy measure ν. Then X_t can be written as

$$X(t) = \int_{0<|x|<1} xq(t, dx) + \int_{|x|\geq 1} xN(t, dx) \tag{A.3}$$

where, N is a Poisson random measure with mean function $\tilde{\nu}((0, t] \times D) = t\nu(D)$ for any measurable set D of $\mathbb{R}_0 = \mathbb{R} \setminus \{0\}$ (see [1]).

Then we can write diffusion equation (2.1) as:

$$dY_t = a(Y_t)dt + \int_{0<|x|<1} b(Y_t)xq(dt, dx) + \int_{|x|\geq 1} b(Y_t)xN(dt, dx). \tag{A.4}$$

The theorem is proved in two steps. In the first step is to consider equation (A.4) up to small jumps only, i.e consider

$$dZ_t = a(Z_t)dt + \int_{0<|x|<1} b(Z_t)xq(dt, dx)$$

such a process $\{Z_t\}$ can be constructed using *Picard's* method (Skorokhod [20]). Further it can be shown the solution $\{Z_t\}$ is unique.

In the second step the large jumps can be added to $\{Z_t\}$ using the interlacing technique (Ikeda and Watanabe [13]) and hence the solution process $\{Y_t\}$ for the main equation (A.4) is constructed. Uniqueness is followed from uniqueness of $\{Z_t\}$ and the interlacing structure.

For detailed construction of the solution process and proof of uniqueness see [1] (Theorems 6.2.3 and 6.2.9). For the proof of continuity w.r.t initial value see [20]. □

Time Homogeneity of the Solution Process $\{Y_t\}$

Let us consider the SDE in (2.1) with a little modification. We assume the coefficient functions a and b satisfy growth condition and Lipschitz condition from Theorem A.1. Consider

$$dY_t = a(Y_t)dt + b(Y_t)dX_t \tag{B.1}$$

for, $t \geq s$, given $Y_s = y$ and X_s is a jump Lévy process.

Denote the unique solution of (B.1) by $Y_t = Y_t^{y,s}$. Then,

$$Y_{t+h}^{y,t} = y + \int_t^{t+h} a\left(Y_u^{y,t}\right) du + \int_t^{t+h} b\left(Y_u^{y,t}\right) dX_u$$
$$= y + \int_0^h a\left(Y_{t+v}^{y,t}\right) dv + \int_0^h b\left(Y_{t+v}^{y,t}\right) d\tilde{X}_v$$

with $u = t + v$ and $\tilde{X}_v = X_{t+v} - X_t$.

On the other hand,

$$Y_h^{y,0} = y + \int_0^h a\left(Y_v^{y,0}\right) dv + \int_0^t b\left(Y_v^{y,0}\right) dX_v.$$

Now, from the definition of Lévy processes, $\{\tilde{X}_v\}$ and $\{X_v\}$ have same X_0-distribution. Then it follows from the weak uniqueness of the solution for SDE in (B.1) that $\{Y_{t+h}^{y,t}\}_{h\geq 0}$ and $\{Y_h^{y,0}\}$, $h \geq 0$ have same Y_0-distribution; i.e, $\{Y_t\}_{t\geq 0}$ is time homogeneous.

Markov Property of the Solution Process $\{Y_t\}$

Let $\{\mathcal{F}_t\}$ be the σ-field generated by $\{X_s : s \leq t\}$. Let $\{Y_t\}$ be the solution process of (2.1) then Y_t is $\mathcal{F}_t$ measurable.

Theorem C.1 *If there exists a unique solution of SDE* (2.1), *then the solution process is a Markov process.*

Proof From construction we can re-write Y_{t+s} as follows:

$$Y_{t+s} = Y_s + M(s, t+s)$$

where,

$$M(s, t+s) = \int_s^{t+s} a(Y_v)dv + \int_s^{t+s} b(Y_v)dX_v.$$

Following the Lévy-Itô decomposition $M(s, t+s)$ is $\sigma\{N(v, A) - N(u, A), s \leq u < v \leq t, A \in \mathcal{B}(\mathbb{R}_0)\}$ measurable ($\mathcal{B}(\mathbb{R}_0)$ is the Borel sigma field of $\mathbb{R}_0$). Clearly $M(s, t+s)$ is independent of $\mathcal{F}_s \subseteq \sigma\{N(u, A), u \leq s, A \in \mathcal{B}(\mathbb{R}_0\}$ and that leads to the Markov property. See [20] (page 75, Theorem 1) for a more detailed proof. □

C.1 Derivation of the Infinitesimal Generator and the Backward Equation of the SDE Led by a Lévy Jump Process

Proof of Theorem 2.1 Let us assume $Y_0 = y$. Using Itô formula for Lévy type integral ([13], Theorem 4.1) for any real valued, twice differentiable function f with compact support, we have,

$$f(Y_t) - f(y)$$
$$= \int_0^t f'(Y_s)a(Y_s)ds + \int_0^t \int_{|x|\geq 1} \left\{f\left(Y(s-) + b(Y_s)x\right) - f\left(Y(s-)\right)\right\} N(ds, dx)$$

$$+\int_0^t \int_{0<|x|<1} \{f(Y(s-)+b(Y_s)x) - f(Y(s-))\} q(ds,dx)$$
$$+\int_0^t \int_{0<|x|<1} \{f(Y(s-)+b(Y_s)x) - f(Y(s-)) - b(Y_s)xf'(Y(s-))\} \tilde{\nu}(ds,dx).$$

By definition of infinitesimal generator we have:

$$\begin{aligned}
Af(y) &= \lim_{t\downarrow 0} \frac{E^y[f(Y_t)] - f(y)}{t} \\
&= \lim_{t\downarrow 0} \frac{1}{t} \int_0^t E^y\big[f'(Y_s)a(Y_s)\big]ds \\
&\quad + \lim_{t\downarrow 0} \frac{1}{t} \int_0^t \int_{|x|\geq 1} E^y\big[f(Y(s-)+b(Y_s)x) - f(Y(s-))\big] ds\,\nu(dx) \\
&\quad + \lim_{t\downarrow 0} \frac{1}{t} \int_0^t \int_{0<|x|<1} E^y\big[f(Y(s-)+b(Y_s)x) - f(Y(s-)) \\
&\quad - b(Y_s)xf'(Y(s-))\big] ds\,\nu(dx) \\
&= f'(y)a(y) + \int_{|x|\geq 1} \{f(y+b(y)x) - f(y)\}\nu(dx) \\
&\quad + \int_{0<|x|<1} \{f(y+b(y)x) - f(y) - b(y)xf'(y)\}\nu(dx) \\
&= f'(y)\big(a(y) - K_\nu b(y)\big) + \int_{\mathbb{R}_0} \{f(y+b(y)x) - f(y)\}\nu(dx) \\
\text{or} \quad & f'(y)\big(a(y) + C_\nu b(y)\big) + \int_{\mathbb{R}_0} \{f(y+b(y)x) - f(y) - f'(y)b(y)x\}\nu(dx)
\end{aligned}$$

where, $K_\nu = \int_{0<|x|<1} x\nu(dx)$ and $C_\nu = \int_{|x|\geq 1} x\nu(dx)$, whenever they are defined. Hence the theorem. □

Proof of Theorem 2.2 Let $g(x) = u(x,t)$. Then, using Markov property we get the following:

$$\begin{aligned}
\frac{E^y[g(Y_r)] - g(y)}{r} &= \frac{1}{r}\left\{E^y\left[E^{Y_r}(g(Y_t))\right] - E^y\left[g(Y_t)\right]\right\} \\
&= \frac{1}{r}\left\{E^y\left[E^y(g(Y_{t+r})|\mathcal{F}_r)\right] - E^y\left[g(Y_t)\right]\right\} \\
&= \frac{1}{r}E^y\left[g(Y_{t+r}) - g(Y_t)\right] \\
&= \frac{u(y,t+r) - u(y,t)}{r} \to \frac{\partial u}{\partial t} \quad \text{as } r \downarrow 0.
\end{aligned}$$

Therefore, $A(u) = \lim_{r\downarrow 0} \frac{E^y[g(Y_r)]-g(y)}{r}$ exists, and $\frac{\partial u}{\partial t} = A(u)$. This gives the backward equation. □

Appendix B

D.1 Pseudo Differential Operator Form of the Infinitesimal Generator and Derivation of the Forward Equation

In this section we shall show that the infinitesimal generator is a pseudo-differential operator (in the sense of Jacob [14]). Using this form with the backward equation we shall get the forward equation by involution type technique. A pseudo differential operator is defined using Fourier transform and negative definite functions. For this section we are going to use Fourier transform of a real function f as follows:

$$\hat{f}(\xi) = F\big(f(\xi)\big) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{-ix\xi} f(x)dx.$$

We shall discuss other required concepts and definitions as we proceed.

Lemma D.1 *Let $\{Y_t\}$ be the solution process of the SDE in (2.1). Let A be the infinitesimal generator for $\{Y_t\}$, given as in Theorem 2.1. Let us make a change of variable $b(y)x = -v$ in (2.2). Let $J(v)$ be the Jacobian of the transformation. We define $\nu_1(y, dv) = J(v)\nu(\frac{-dv}{b(y)})$. Then we can show A can be expressed as follows*:

$$Af(y) = B(y)f'(y) + \int_{\mathbb{R}_0} \left[f(y-v) - f(y) + f'(y)\frac{v}{1+|v|^2} \right] \nu_1(y, dv) \tag{D.1}$$

where,

$$B(y) = \left[a(y) + C_\nu b(y) + \int_{\mathbb{R}_0} v\left(\frac{|v|^2}{1+|v|^2}\right)\nu_1(y, dv)\right]; \quad \text{if } C_\nu < \infty.$$

And,

$$B(y) = \left[a(y) - K_\nu b(y) - \int_{\mathbb{R}_0} \left(\frac{v}{1+|v|^2}\right)\nu_1(y, dv)\right]; \quad \text{if } K_\nu < \infty.$$

Proof From Theorem 2.1, if we consider form (2.2a), the infinitesimal generator can be written as:

$$Af(y) = f'(y)\big(a(y) + C_\nu b(y)\big) + \int_{\mathbb{R}_0} \big\{ f\big(y + b(y)x\big) - f(y) - f'(y)b(y)x \big\} \nu(dx).$$

Let us consider the change of variable $b(y)x = -v$. Let $J(v)$ be the Jacobian of transformation we define : $\nu_1(y, dv) = J(v)\nu(\frac{-dv}{b(y)})$. Then

$$\begin{aligned} Af(y) &= f'(y)\big(a(y) + C_\nu b(y)\big) + \int_{\mathbb{R}_0} \big\{ f(y-v) - f(y) + f'(y)v \big\} \nu_1(y, dv) \\ &= f'(y)\left[a(y) + C_\nu b(y) + \int_{\mathbb{R}_0} v\left(\frac{|v|^2}{1+|v|^2}\right)\nu_1(y, dv)\right] \\ &\quad + \int_{\mathbb{R}_0} \left\{ f(y-v) - f(y) + f'(y)\frac{v}{1+|v|^2} \right\} \nu_1(y, dv) \end{aligned}$$

$$= B(y)f'(y) + \int_{\mathbb{R}_0} \left[f(y-v) - f(y) + f'(y)\frac{v}{1+|v|^2} \right] \nu_1(y, dv)$$

where,

$$B(y) = \left[a(y) + C_\nu b(y) + \int_{\mathbb{R}_0} v \left(\frac{|v|^2}{1+|v|^2} \right) \nu_1(y, dv) \right].$$

From Theorem 2.1, if we consider form (2.2b), the infinitesimal generator can be written as

$$Af(y) = f'(y)\big(a(y) - K_\nu b(y)\big) + \int_{\mathbb{R}_0} \big\{ f\big(y + b(y)x\big) - f(y) \big\} \nu(dx).$$

Again change of variable $b(y)x = -v$ gives,

$$Af(y) = f'(y)\big(a(y) - K_\nu b(y)\big) + \int_{\mathbb{R}_0} \big\{ f(y-v) - f(y) \big\} \nu_1(y, dv)$$

$$= B(y)f'(y) + \int_{\mathbb{R}_0} \left[f(y-v) - f(y) + f'(y)\frac{v}{1+|v|^2} \right] \nu_1(y, dv)$$

where,

$$B(y) = \left[a(y) - K_\nu b(y) - \int_{\mathbb{R}_0} \left(\frac{v}{1+|v|^2} \right) \nu_1(y, dv) \right].$$

Hence the lemma is proved. □

Next, we give some definitions from Analysis. We shall be using following notations consistent with [14]:

$C^\infty(\mathbb{R})$:= class of arbitrarily many times differentiable functions on $\mathbb{R}$.
$C_\infty(\mathbb{R})$:= class of continuous functions on $\mathbb{R}$ vanishing at ∞.
$\mathbb{N}_0 := \{0, 1, 2, \ldots\}$.

Definition D.1 (Lévy Khinchin representation) We say $\psi : \mathbb{R} \to \mathbb{C}$ is a continuous negative definite function, if ψ has the following representation:

$$\psi(\xi) = c + i(d\xi) + \tilde{q}(\xi) + \int_{\mathbb{R}_0} \left(1 - e^{-ix\xi} - \frac{ix\xi}{1+|x|^2} \right) \nu(dx). \tag{D.2}$$

With $c > 0$, $d \in \mathbb{R}$, $\tilde{q}$ is symmetric positive semi-definite quadratic form on $\mathbb{R}$ and ν is the Lévy measure associated with ψ such that,

$$\int_{\mathbb{R}_0} (|x|^2 \wedge 1)\nu(dx) < \infty.$$

ψ is uniquely determined by $(c, d, \tilde{q}, \nu)$.

Definition D.2 We call a function $Q : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ a *continuous negative definite symbol* if Q is locally bounded and for each $x \in \mathbb{R}$ the function $Q(x, .) : \mathbb{R} \to \mathbb{C}$ is continuous negative definite.

Definition D.3 We define *Schwartz space* $\mathcal{S}(\mathbb{R})$ consist of all function $u \in C^{\infty}(\mathbb{R})$ such that for all $m_1 m_2 \in \mathbb{N}_0$

$$p_{m_1,m_2}(u) := \sup_{x\in\mathbb{R}}\left[(1+|x|^2)^{m_1/2}\sum_{k\le m_2}|\partial^k u(x)|\right] < \infty.$$

The pseudo-differential operators associated with the symbol $Q(x,\xi)$ are defined as follows:

Definition D.4 For a continuous negative definite symbol $Q(x,\xi)$, we define the pseudo-differential operator $Q(x,D)$ by :

$$Q(x,D)u(x) := (2\pi)^{-1/2}\int_{\mathbb{R}} e^{ix\xi} Q(x,\xi)\hat{u}(\xi)d\xi \tag{D.3}$$

for $u \in \mathcal{S}(\mathbb{R})$.

The next theorem gives the pseudo-differential operator representation of the infinitesimal generator.

Theorem D.1 *Let us use the measure* ν_1 *and coefficient function* B *from Lemma* D.1 *to define following continuous negative definite symbol*:

$$Q(x,\xi) = \int_{\mathbb{R}_0}\left(1 - e^{-iv\xi} - \frac{iv\xi}{1+|v|^2}\right)\nu_1(x,dv) - iB(x)\xi. \tag{D.4}$$

Let A *be the infinitesimal generator defined in Sect.* 2. *Then the infinitesimal generator, restricted in* $\mathcal{S}(\mathbb{R})$ *is a pseudo-differential operator with negative definite symbol* Q *as above. That is*:

$$Af(x) = -Q(x,D)f(x) = -(2\pi)^{-\frac{1}{2}}\int e^{ix\xi}Q(x,\xi)\hat{f}(\xi)d\xi \tag{D.5}$$

where, $\hat{f}$ *is the Fourier transformation of* $f \in \mathcal{S}(\mathbb{R})$.

Proof We shall use the notation $\hat{f} = F(f)$ to denote the Fourier transform as defined in the beginning of this section. Also recall that, the inverse Fourier transform of a function g is defined as

$$F^{-1}g(\eta) = (2\pi)^{-1/2}\int_{-\infty}^{\infty} e^{ix\eta}g(x)dx.$$

Then, using Lemma D.1,

$$\begin{aligned}
-Q(x,D)f &= -(2\pi)^{-\frac{1}{2}}\int e^{ix\xi}Q(x,\xi)\hat{f}(\xi)d\xi \\
&= -(2\pi)^{-\frac{1}{2}}\int e^{ix\xi}\left[\int_{\mathbb{R}_0}\left(1-e^{-iv\xi}-\frac{iv\xi}{1+|v|^2}\right)\nu_1(x,dv) - iB(x)\xi\right]\hat{f}(\xi)d\xi \\
&= -\int_{\mathbb{R}_0}\left[(2\pi)^{-\frac{1}{2}}\int e^{ix\xi}F\left(f(\xi)-f(\xi-v)-\frac{v}{1+|v|^2}f'(\xi)\right)d\xi\right]\nu_1(x,dv)
\end{aligned}$$

$$+ B(x)\left[(2\pi)^{-\frac{1}{2}} \int e^{ix\xi} F\big(f'(\xi)\big) d\xi\right]$$

$$= -\int_{\mathbb{R}_0} \left[f(x) - f(x-v) - f'(x)\frac{v}{1+|v|^2}\right]\nu_1(x,dv) + B(x)f'(x)$$

$$= Af(x).$$

Hence, Theorem D.1 is proved. □

Next, we shall discuss the spaces associated with the pseudo differential operators. For a real valued continuous negative definite function $\psi(\xi)$ and $s \geq 0$ let us define a norm

$$\|u\|^2_{\psi,s} := \int_{\mathbb{R}} [1+\psi(\xi)]^s |\hat{u}(\xi)|^2 d\xi.$$

An anisotropic Sobolev space with a negative definite function ψ is given by

$$H^{\psi,s}(\mathbb{R}) := \{R \in L^2(\mathbb{R}) : \|u\|^2_{\psi,s} < \infty\}.$$

These are Hilbert spaces under the norm $\|u\|^2_{\psi,s}$ and arise naturally in the discussion of the pseudo-differential operators (see [12, 15]). Jacob and Schilling [15] showed that with appropriate choice of ψ, a pseudo-differential $Q(x,D)$ operator associated with the generator of a Lévy type processes maps the space $H^{\psi,s+2}$ to $H^{\psi,s}$ and hence using Sobolev's embedding theorem, $Q(x,D)$ can be extended to $C_\infty(\mathbb{R})$. Thus we can say the pseudo differential operator presentation of infinitesimal generator A can be extended to $C_\infty(\mathbb{R})$.

In our case we shall use the Sobolev space $H^{\xi^2,2}$ and we shall assume the transition density of the solution process belongs to this space. Most of the cases the density of a Lévy type processes belongs to this class, for example use the stable characteristic function (Definition E.1) to see that the stable-Lévy density belongs to $H^{\xi^2,2}$.

We shall use the following bound of symbol (D.4) in the derivation of the forward equation.

Lemma D.2 *Consider an SDE of form* (B.1). *For the coefficient functions a and b let us define*

$M(x) := \max\{|a(x)|, |b(x)|^2\}$. *For the continuous negative definite symbol* $Q(x,\xi)$ *given in* (D.4) *if the Lévy measure* ν *of* Q *satisfies* $\int_{|y|>1} \nu(dy) < \infty$. *Then,*

$$|Q(x,\xi)| \leq cM(x)(1+\xi^2) \tag{D.6}$$

for some constant c.

Proof Consider $Q(x,\xi)$ given in (D.4). First, consider the form $B(x) = a(x) + C_\nu b(x) + \int_{\mathbb{R}_0} v \frac{|v|^2}{1+|v|^2} \nu_1(y,dv)$

$$Q(x,\xi) = \int_{\mathbb{R}_0} \left(1 - e^{-iv\xi} - \frac{iv\xi}{1+|v|^2}\right)\nu_1(x,dv) - iB(x)\xi$$

$$= \int_{\mathbb{R}_0} \big(1 - e^{-iv\xi} - iv\xi\big)\nu_1(x,dv) - i\big[a(x) + C_\nu b(x)\big]\xi.$$

Using reverse transformation $b(x)y = -v$

$$Q(x,\xi) = \int_{\mathbb{R}_0} \big(1 - e^{ib(x)y\xi} + ib(x)y\xi\big)\nu(dy) - i\big[a(x) + C_\nu b(x)\big]\xi$$
$$= \int_{0<|y|<1} \big(1 - e^{ib(x)y\xi} + ib(x)y\xi\big)\nu(dy) + \int_{|y|\geq 1} \big(1 - e^{ib(x)y\xi}\big)\nu(dy) - i\xi a(x).$$

Next, consider the form $B(x) = a(x) - K_\nu b(x) - \int_{\mathbb{R}_0} \frac{v}{1+|v|^2}\nu_1(y, dv)$.

$$Q(x,\xi) = \int_{\mathbb{R}_0} \left(1 - e^{-iv\xi} - \frac{iv\xi}{1+|v|^2}\right)\nu_1(x, dv) - iB(x)\xi$$
$$= \int_{\mathbb{R}_0} \big(1 - e^{-iv\xi}\big)\nu_1(x, dv) - i\big[a(x) - K_\nu b(x)\big]\xi.$$

Using reverse transformation $-b(x)y = v$ as the previous case,

$$Q(x,\xi) = \int_{\mathbb{R}_0} \big(1 - e^{ib(x)y\xi}\big)\nu(dy) - i\big[a(x) - K_\nu b(x)\big]\xi$$
$$= \int_{0<|y|<1} \big(1 - e^{ib(x)y\xi} + ib(x)y\xi\big)\nu(dy) + \int_{|y|\geq 1} \big(1 - e^{ib(x)y\xi}\big)\nu(dy) - i\xi a(x).$$

Thus, for both forms of $B(\cdot)$ we can write:

$$Q(x,\xi) = \int_{0<|y|<1} \big(1 - e^{ib(x)y\xi} + ib(x)y\xi\big)\nu(dy) + \int_{|y|\geq 1} \big(1 - e^{ib(x)y\xi}\big)\nu(dy) - i\xi a(x) \quad \text{(D.7)}$$

further, note that we can get the following bounds :

$$\big|1 - e^{ib(x)y\xi} + ib(x)y\xi\big| \leq |b(x)\xi y|^2 \quad \text{and} \quad |1 - e^{ib(x)y\xi}| \leq 2$$

also, $|\xi| \leq (1 + |\xi|^2)$. Recall that ν, the Lévy measure for $Q(x, D)$ in (D.4), satisfies $\int_{0<|y|<1} |y|^2\nu(dy) < \infty$. Therefore, if we assume $\int_{|y|\geq 1} \nu(dy) < \infty$, then

$$|Q(x,\xi)| \leq |\xi|^2|b(x)|^2 \int_{0<|y|<1} |y|^2\nu(dy) + \int_{|y|\geq 1} 2\nu(dy) + |a(x)||\xi|$$
$$\leq cM(x)(1 + |\xi|^2)$$

where $M(x) = \max\{|b(x)|^2, a(x)\}$ and c is a constant that depends only on ν. □

Using this bound we can prove Theorem 2.3.

Proof of Theorem 2.3 Let $p_x(t, y)$ be the transition probability density of Y_t starting at $Y_0 = x$. Let $u_0 : \mathbb{R} \to \mathbb{R}$ be a twice differentiable bounded function with finite second moment. Let us write $u(x, t) = E^x[u_0(Y_t)] = \int_{\mathbb{R}} u_0(y)p_x(t, y)dy$.

Then $Au(x, t) = \frac{\partial}{\partial t}u(x, t)$ is defined. And some constant c', such that,

$$|\hat{u}(\xi, t)|^2 \leq c'|\hat{p}_0(t, \xi)|^2. \quad \text{(D.8)}$$

Since the transition density function vanishes at $t=\infty$, then integration by parts gives

$$\int_{-\infty}^{\infty}\int_{0}^{\infty}\frac{\partial}{\partial t}u(y,t)p_x(t,y)dtdy=-\int_{-\infty}^{\infty}\int_{0}^{\infty}u(y,t)\frac{\partial}{\partial t}p_x(t,y)dtdy. \tag{D.9}$$

Now, substituting the backward equation in the left hand side we get,

$$\int_{-\infty}^{\infty}\int_{0}^{\infty}p_x(t,y)A\big(u(y,t)\big)dtdy=-\int_{-\infty}^{\infty}\int_{0}^{\infty}u(y,t)\frac{\partial}{\partial t}p_x(t,y)dtdy. \tag{D.10}$$

The operator A acts on *u(y,t)* as u being a functions of y. Consider the integration part with respect to y in the left hand side of (D.10) and to simplify the notations we ignore the other variables in the term for next steps of computation and write:

$$\int_{-\infty}^{\infty}p_x(t,y)A\big(u(y,t)\big)dy=\int p(y)Au(y)dy.$$

Using Cauchy-Schwarz inequality,

$$\left|\int p(y)Au(y)dy\right|^2\le\int|p(y)|^2dy\int|Au(y)|^2dy.$$

Parseval identity gives

$$\int|p(y)|^2dy=\int|\hat{p}(\xi)|^2d\xi\ \le\ \int(1+\xi^2)|\hat{p}(\xi)|^2d\xi=\|v\|^2_{\xi^2,2}.$$

Whereas, pseudo differential operator form of the generator, equation (D.8) and Lemma D.2 gives

$$\begin{aligned}\int|Au(y)|^2dy&=(2\pi)^{-1}\int\left|\int e^{iy\xi}Q(y,\xi)\hat{u}(\xi)d\xi\right|^2dy\\&\le(2\pi)^{-1}\int\int|Q(y,\xi)|^2|\hat{u}(\xi)|^2d\xi dy\\&\le\int M^2(y)\int(1+\xi^2)|\hat{u}(\xi)|^2d\xi dy\\&=c'\|p\|^2_{\xi^2,2}\int M^2(y)dy.\end{aligned}$$

By hypothesis $\int M^2(y)dy<\infty$, therefore for $p\in H^{\xi^2,2}$ we can say $|\int p(y)Au(y)dy|^2<\infty$. Thus we can use Fubini theorem for $\int p(y)Au(y)dy$.

Also for the negative definite symbol Q given in (D.4) the following holds:

$$Q(x,\xi)=\int_{\mathbb{R}_0}\left(1-e^{-iv\xi}-\frac{iv\xi}{1+|v|^2}\right)\nu_1(x,dv)-iB(x)\xi.$$

If we use $B(x)=a(x)+C_\nu b(x)+\int_{\mathbb{R}_0}v\frac{v^2}{1+v^2}\nu_1(x,dv)$

$$Q(x,\xi)=\int_{\mathbb{R}_0}(1-e^{-iv\xi}-iv\xi)\nu_1(x,dv)-i\xi\big[a(x)+C_\nu b(x)\big].$$

We write,

$$Q(x,\xi) = \int_{\mathbb{R}_0} \big(1 - e^{-iv\xi} - iv\xi\big)\nu_1(x,dv) - iG(x)\xi$$
$$\text{where } G(x) = \big[a(x) + C_\nu b(x)\big]. \tag{D.11a}$$

If we use $B(x) = a(x) - K_\nu b(x) - \int_{\mathbb{R}_0} \frac{v}{1+v^2}\nu_1(x,dv)$

$$Q(x,\xi) = \int_{\mathbb{R}_0} (1 - e^{-iv\xi})\nu_1(x,dv) - i\xi\big[a(x) - K_\nu b(x)\big].$$

We write,

$$Q(x,\xi) = \int_{\mathbb{R}_0} \big(1 - e^{-iv\xi}\big)\nu_1(x,dv) - iH(x)\xi$$
$$\text{where } H(x) = \big[a(x) - K_\nu b(x)\big]. \tag{D.11b}$$

In order to get a closed form of the forward equation, we need to assume ν_1 has the form $\nu_1(x,dy) = h(x)\mu(dy)$, here h is real valued function and μ is a measure on $\mathbb{R}$. Using the pseudo differential representation of A and Fubini theorem we get:

$$\begin{aligned}\int_{-\infty}^{\infty} p(x)Au(x)dx &= -(2\pi)^{-1/2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{ix\xi}Q(x,\xi)\hat{u}(\xi)p(x)d\xi dx\\ &= -(2\pi)^{-1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{ix\xi}e^{-ix'\xi}Q(x,\xi)u(x')p(x)d\xi dx dx'\\ &= \int_{-\infty}^{\infty}\left[-(2\pi)^{-1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{ix\xi}e^{-ix'\xi}Q(x,\xi)p(x)d\xi dx\right]u(x')dx'\\ &= \int_{-\infty}^{\infty} Ip(x')u(x')dx' \end{aligned} \tag{D.12}$$

where,

$$Ip(x') = \left[-(2\pi)^{-1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{ix\xi}e^{-ix'\xi}Q(x,\xi)p(x)d\xi dx\right] \tag{D.13}$$

using $Q(x,\xi)$ from (D.11a),

$$\begin{aligned} Ip(x') &= -(2\pi)^{-1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{ix\xi}e^{-ix'\xi}\left[\int_{\mathbb{R}_0}\big(1 - e^{-iv\xi} - iv\xi\big)\nu_1(x,dv) - iG(x)\xi\right]\\ &\quad\times p(x)d\xi dx\\ &= I_1 + I_2 \end{aligned}$$

$$\text{where } I_1 = -(2\pi)^{-1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{\mathbb{R}_0} e^{ix\xi}e^{-ix'\xi}\big(1 - e^{-iv\xi} - iv\xi\big)p(x)\nu_1(x,dv)d\xi dx,$$

$$I_2 = (2\pi)^{-1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{ix\xi}e^{-ix'\xi}iG(x)\xi p(x)d\xi.$$

For the first part we have,

$$I_1 = -(2\pi)^{-1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{\mathbb{R}_0} e^{ix\xi}e^{-ix'\xi}\big(1 - e^{-iy\xi} - iy\xi\big)p(x)\nu_1(x,dy)d\xi dx$$

$$= -(2\pi)^{-1} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{\mathbb{R}_0} e^{ix\xi} e^{-ix'\xi} \big(1 - e^{-iy\xi} - iy\xi\big) p(x) h(x) \mu(dy) d\xi dx$$

$$= -(2\pi)^{-1/2} \int_{-\infty}^{\infty} \int_{\mathbb{R}_0} e^{-ix'\xi} \big(1 - e^{-iy\xi} - iy\xi\big) F\big((p \cdot h)(-\xi)\big) \mu(dy) d\xi$$

$$= -(2\pi)^{-1/2} \int_{-\infty}^{\infty} \int_{\mathbb{R}_0} e^{ix'\xi} \big(1 - e^{iy\xi} + iy\xi\big) F\big((p \cdot h)(\xi)\big) \mu(dy) d\xi$$

$$= -(2\pi)^{-1/2} \int_{\mathbb{R}_0} \int_{-\infty}^{\infty} e^{ix'\xi} F\big[(p \cdot h)(\xi) - (p \cdot h)(\xi + y) + y(p \cdot h)'(\xi)\big] d\xi \mu(dy)$$

$$= -\int_{\mathbb{R}_0} F^{-1} \circ F\big[(p \cdot h)(x') - (p \cdot h)(x' + y) + y(p \cdot h)'(x')\big] \mu(dy)$$

$$= \int_{\mathbb{R}_0} \big[(p \cdot h)(x' + y) - (p \cdot h)(x') - y(p \cdot h)'(x')\big] \mu(dy)$$

$$= -\int_{\mathbb{R}_0} \big[(p \cdot h)(x' - y) - (p \cdot h)(x') + y(p \cdot h)'(x')\big] \mu(d(-y)).$$

For the other part we have,

$$I_2 = (2\pi)^{-1} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-ix'\xi} e^{ix\xi} i\xi G(x) p(x) d\xi dx$$

$$= (2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{-ix'\xi} i\xi F\big[(p \cdot G)(-\xi)\big] d\xi$$

$$= (2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{ix'\xi'} F\big[(p \cdot G)'(\xi')\big] d\xi'$$

$$= F^{-1} \circ F(p \cdot G)'(x') = (p \cdot G)'(x').$$

Thus, in case we use the negative definite symbol from (D.11a), we can write

$$Ip(x') = -\bigg[\int_{\mathbb{R}_0} \big[(p \cdot h)(x' - y) - (p \cdot h)(x') + y(p \cdot h)'(x')\big] - (G \cdot p)'(x')\bigg] \quad \text{(D.14a)}$$

Using $Q(x, \xi)$ of the from (D.11b) in (D.14) we have, $Ip(x') = I_3 + I_4$. Where,

$$I_3 = -(2\pi)^{-1} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{\mathbb{R}_0} e^{ix\xi} e^{-ix'\xi} \big(1 - e^{-iv\xi}\big) p(x) \nu_1(x, dv) d\xi dx,$$

$$I_4 = (2\pi)^{-1} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{ix\xi} e^{-ix'\xi} i H(x) \xi p(x) d\xi dx.$$

Using similar derivations as I_1 and I_2, it can be shown

$$I_3 = -\int_{\mathbb{R}_0} \big[(p \cdot h)(x' - y) - (p \cdot h)(x')\big] \mu(d(-y)).$$

And for the next integration term it can be shown, $I_4 = (p \cdot H)'(x')$. Therefore, in case we use the negative definite symbol from (D.11b), we can write:

$$Ip(x') = -\left[\int_{\mathbb{R}_0} \left[(p \cdot h)(x' - y) - (p \cdot h)(x')\right]\mu(d(-y)) - (H \cdot p)'(x')\right] \tag{D.14b}$$

then from (D.12) and (D.10) we have,

$$\int_{-\infty}^{\infty}\int_0^{\infty} u(y,t)Ip_x(t,y)dtdy = -\int_{-\infty}^{\infty}\int_0^{\infty} u(y,t)\frac{\partial}{\partial t}p_x(t,y)dtdy$$
$$\Rightarrow \quad \int_{-\infty}^{\infty}\int_0^{\infty}\left[Ip_x(t,y) + \frac{\partial}{\partial t}p_x(t,y)\right]u(y,t)dtdy = 0.$$

Note that, this is true for any arbitrary choice of twice differentiable bounded $u_0(\cdot)$. Hence we must have $[Ip_x(t,y) + \frac{\partial}{\partial t}p_x(t,y)] = 0$, thus $\frac{\partial}{\partial t}p_x(t,y) = -Ip_x(t,y)$. Then, combining (D.14a) and (D.14b), the forward equation is given as follows:

$$\frac{\partial}{\partial s}p_x(s,y) = \int_{\mathbb{R}_0}\left[(p_x \cdot h)(s, y - r) - (p_x \cdot h)(s,y) + r(p_x \cdot h)'(s,y)\right]\mu(d(-r))$$
$$- \frac{\partial}{\partial y}(p_x \cdot G)(s,y)$$
$$\text{where, } (p_x \cdot h)(s,y) = h(y)p_x(s,y) \quad \text{and } G(y) = a(y) + C_\nu b(y);$$
$$\text{in case } C_\nu < \infty. \tag{D.15a}$$

Alternatively,

$$\frac{\partial}{\partial s}p_x(s,y) = \int_{\mathbb{R}_0}\left[(p_x \cdot h)(s, y - r) - (p_x \cdot h)(s,y)\right]\mu(d(-r)) - \frac{\partial}{\partial y}(p_x \cdot H)(s,y)$$
$$\text{where, } (p_x \cdot h)(s,y) = h(y)p_x(s,y) \quad \text{and } H(y) = a(y) - K_\nu b(y);$$
$$\text{in case } K_\nu < \infty. \tag{D.15b}$$

That concludes the theorem. □

Remark The forward equation theorem can be used to solve an interesting analytical problem. Let $\mathcal{S}'(\mathbb{R})$ be the space of tempered distribution which is the dual space of $\mathcal{S}(\mathbb{R})$. If we assume $u(y,t)$ in $\mathcal{S}(\mathbb{R})$, then $Au(y,t) = -Q(y,D)u$ belongs to $\mathcal{S}(\mathbb{R})$. The solution of the forward equation actually produces an element in $\mathcal{S}'(\mathbb{R})$, which is the solution of the adjoint operator (forward operator) given by the transition function.

Appendix C

E.1 Derivation of the Infinitesimal Generator, the Backward and the Forward Equation for the SDE Driven by a Stable Process

Let us consider a special Lévy process: an α-*stable Lévy process*. First let us define a *stable distribution*:

Definition E.1 A random variable X is said to have a stable distribution with index of stability α, scale parameter σ, skewness parameter β and the shift parameter μ if its characteristic function has the following form:

$$E[\exp i\theta X] = \begin{cases} \exp\{-\sigma^\alpha|\theta|^\alpha(1 - i\beta(\operatorname{sign}\theta)\tan(\frac{\pi\alpha}{2})) + i\mu\theta\}, & \text{if } \alpha \neq 1, \\ \exp\{-\sigma|\theta|(1 + i\beta\frac{2}{\pi}(\operatorname{sign}\theta)\ln|\theta|) + i\mu\theta\}, & \text{if } \alpha = 1 \end{cases}$$

where $0 < \alpha \leq 2$, $\sigma \geq 0$, $-1 \leq \beta \leq 1$, μ real and

$$\operatorname{sign}\theta = \begin{cases} 1, & \text{if } \theta > 0, \\ 0, & \text{if } \theta = 0, \\ -1, & \text{if } \theta < 0 \end{cases}$$

we write $X \sim S_\alpha(\sigma, \beta, \mu)$. For an α-stable Lévy process $\{X_t\}$ with skewness parameter β, for $0 \leq s < t < \infty$, $(X_t - X_s) \sim S_\alpha((t-s)^{1/\alpha}, \beta, 0)$.

Next proposition gives the precise form of the fractional derivatives:

Proposition E.1 (See [2] for details) *The fractional derivative of order α for a function f can be expressed as follows*:

(a) *For* $0 < \alpha < 1$,

$$\frac{\partial^\alpha}{\partial x^\alpha} f(x) = \frac{\alpha}{\Gamma(1-\alpha)} \int_0^\infty \big(f(x-v) - f(x)\big)\frac{dv}{|v|^{1+\alpha}}, \tag{E.1}$$

$$\frac{\partial^\alpha}{\partial(-x)^\alpha} f(x) = \frac{\alpha}{\Gamma(1-\alpha)} \int_{-\infty}^0 \big(f(x-v) - f(x)\big)\frac{dv}{|v|^{1+\alpha}}. \tag{E.2}$$

(b) *For* $1 < \alpha < 2$,

$$\frac{\partial^\alpha}{\partial x^\alpha} f(x) = \frac{\alpha(\alpha-1)}{\Gamma(2-\alpha)} \int_0^\infty \big(f(x-v) - f(x) + vf'(x)\big)\frac{dv}{|v|^{1+\alpha}}, \tag{E.3}$$

$$\frac{\partial^\alpha}{\partial(-x)^\alpha} f(x) = \frac{\alpha(\alpha-1)}{\Gamma(2-\alpha)} \int_{-\infty}^0 \big(f(x-v) - f(x) + vf'(x)\big)\frac{dv}{|v|^{1+\alpha}}. \tag{E.4}$$

The next theorem describes the Poisson random measure associated with an α-stable Lévy Process.

Theorem E.1 (Theorem 3.12.2: [17]) *Let N be a Poisson random measure defined on* $[0, \infty) \times \mathbb{R}_0$ *with mean function*

$$n(ds, du) = E[N(ds, du)] = \begin{cases} (1+\beta)ds\frac{du}{|u|^{1+\alpha}}, & \text{if } u > 0, \\ (1-\beta)ds\frac{du}{|u|^{1+\alpha}}, & \text{if } u < 0. \end{cases}$$

$$= I_\beta(u)ds\frac{du}{|u|^{1+\alpha}}$$

$$\textit{where } I_\beta(u) = \begin{cases} (1+\beta), & \text{if } u > 0, \\ (1-\beta), & \text{if } u < 0. \end{cases}$$

Now, if we set β to be the skewness parameter of an α-Stable Lévy process $\{X_t\}$; then, for a random function $f : [0, \infty) \times \Omega \to \mathbb{R}$,

(i) *For* $0 < \alpha < 1$:

$$\int_0^t f dX_s \stackrel{d}{=} C_\alpha \int_0^t \int_{\mathbb{R}_0} f(s) u N(ds, du).$$

(ii) *For* $1 < \alpha < 2$:

$$\int_0^t f dX_s \stackrel{d}{=} C_\alpha \times \lim_{\delta \downarrow 0} \left(\int_0^t \int_{(-\delta,\delta)^c} f(s) u q(ds, du) \right)$$

where, constant C_α is defined as follows:

$$(C_\alpha)^\alpha = \begin{cases} (2\alpha^{-1}(\Gamma(1-\alpha)) \cos \frac{\pi\alpha}{2})^{-1}, & \text{if } 0 < \alpha < 1, \\ (2\frac{\Gamma(2-\alpha)}{\alpha(\alpha-1)}(-\cos \frac{\pi\alpha}{2}))^{-1}, & \text{if } 1 < \alpha < 2. \end{cases}$$

Proof of Theorem 3.1 Let us define a Poisson random measure N as in Theorem E.1.

For $0 < \alpha < 1$ The solution processes $\{Y_t\}$ of SDE (3.1) can be written as:

$$Y_t = Y_0 + \int_0^t a(Y_s) ds + C_\alpha \int_0^t \int_{R_0} b(Y_s) u N(ds, du).$$

Let us define Poisson random measure $N_\alpha(\cdot, \cdot)$ as $N_\alpha(s, u) = N(s, C_\alpha u)$. Let $\tilde{\nu}$ be the mean function of N_α. Then the change of variable gives

$$\tilde{\nu}(ds, du) = n\big(ds, d(C_\alpha u)\big) = I_\beta(u) ds (C_\alpha)^\alpha \frac{du}{|u|^{1+\alpha}} = ds \nu_\alpha(du)$$

where, $\nu_\alpha(du) = I_\beta(u)(C_\alpha)^\alpha \frac{du}{|u|^{1+\alpha}}$; $I_\beta(\cdot)$ defined as in Theorem E.1. We define compensated Poisson random measure q_α by, $q_\alpha(\cdot, \cdot) = N_\alpha(\cdot, \cdot) - \tilde{\nu}(\cdot, \cdot)$. Let us denote by $\tilde{a}$, the function: $\tilde{a}(y) = a(y) + b(y) \int_{0<|u|<1} u \nu_\alpha(du)$. Then we can write:

$$\begin{aligned} Y_t = Y_0 + \int_0^t \tilde{a}(Y_s) ds + \int_0^t \int_{0<|u|<1} b(Y_s) u q_\alpha(ds, du) \\ + \int_0^t \int_{|u|\geq 1} b(Y_s) u N_\alpha(ds, du). \end{aligned} \tag{E.5}$$

This is of the same form as the equation given in (A.4). Also, we have function $\tilde{a}$ in place of a and Poisson random measure N_α. From Theorem 2.1 using the infinitesimal generator as in (2.2b), we can derive the infinitesimal generator in case of α-Stable Lévy process as follows:

$$Af(y) = f'(y)\big(\tilde{a}(y) - K_{\nu_\alpha} b(y)\big) + \int_{\mathbb{R}_0} \big\{ f\big(y + b(y)x\big) - f(y) \big\} \nu_\alpha(dx)$$

here, $K_{\nu_\alpha} = \int_{0<|x|<1} x \nu_\alpha(dx)$. Thus,

$$Af(y) = a(y) f'(y) + (C_\alpha)^\alpha \int_{\mathbb{R}_0} \big\{ f\big(y + b(y)x\big) - f(y) \big\} I_\beta(x) \frac{dx}{|x|^{1+\alpha}}$$

$$= a(y)f'(y) + (C_\alpha)^\alpha b^\alpha(y) \int_{\mathbb{R}_0} \{f(y-v) - f(y)\} I_\beta(-v) \frac{dv}{|v|^{1+\alpha}}$$

$$= a(y)f'(y) + (C_\alpha)^\alpha b^\alpha(y)(1-\beta) \int_0^\infty \{f(y-v) - f(y)\} \frac{dv}{|v|^{1+\alpha}}$$

$$+ (C_\alpha)^\alpha b^\alpha(y)(1+\beta) \int_{-\infty}^0 \{f(y-v) - f(y)\} \frac{dv}{|v|^{1+\alpha}}$$

$$= a(y)f'(y) + (C_\alpha)^\alpha b^\alpha(y)(1-\beta)\left[\frac{\Gamma(\alpha-1)}{\alpha} \frac{d^\alpha}{dy^\alpha} f(y)\right]$$

$$+ (C_\alpha)^\alpha b^\alpha(y)(1+\beta)\left[\frac{\Gamma(\alpha-1)}{\alpha} \frac{d^\alpha}{d(-y)^\alpha} f(y)\right].$$

Hence, for $0 < \alpha < 1$, the infinitesimal generator of the solution process for SDE in (3.1) can be written as:

$$Af(y) = a(y)f'(y) + \left[(1-\beta)\left(2\cos\left(\frac{\pi\alpha}{2}\right)\right)^{-1} b^\alpha(y)\frac{d^\alpha}{dy^\alpha} f(y)\right.$$

$$\left. + \ (1+\beta)\left(2\cos\left(\frac{\pi\alpha}{2}\right)\right)^{-1} b^\alpha(y)\frac{d^\alpha}{d(-y)^\alpha} f(y)\right]. \tag{E.6}$$

For $1 < \alpha < 2$ Again from Theorem E.1, the solution process of the SDE (3.1)

$$Y_t = Y_0 + \int_0^t a(Y_s)ds + C_\alpha \lim_{\delta\downarrow 0} \int_0^t \int_{(-\delta,\delta)^c} b(Y_s)uq(ds,du).$$

Let us denote by $\tilde{a}$, the function, $\tilde{a}(y) = a(y) - b(y)\int_{|u|\geq 1} u\nu_\alpha(du)$. Then, we can write:

$$Y_t = Y_0 + \int_0^t \tilde{a}(Y_s)ds + \int_0^t \int_{0<|u|<1} b(Y_s)uq_\alpha(ds,du)$$

$$+ \int_0^t \int_{|u|\geq 1} b(Y_s)uN_\alpha(ds,du) \tag{E.7}$$

this is of the same form as the equation given in (A.4). Also we have function $\tilde{a}$ in place of a and the Poisson random measure N_α. From Theorem 2.1 using the infinitesimal generator as in (2.2a), we can derive the infinitesimal generator in case of α-Stable Lévy process as follows:

$$Af(y) = f'(y)\big(\tilde{a}(y) + C_{\nu_\alpha} b(y)\big) + \int_{\mathbb{R}_0} \{f(y+b(y)x) - f(y) - f'(y)b(y)x\} \nu_\alpha(dx)$$

here, $C_{\nu_\alpha} = \int_{|x|\geq 1} x\nu_\alpha(dx)$. Therefore,

$$Af(y) = a(y)f'(y) + (C_\alpha)^\alpha \int_{\mathbb{R}_0} \{f(y+b(y)x) - f(y) - f'(y)b(y)x\} I_\beta(x)\frac{dx}{|x|^{1+\alpha}}$$

$$= a(y)f'(y) + (C_\alpha)^\alpha b^\alpha(y) \int_{\mathbb{R}_0} \{f(y-v) - f(y) + f'(y)v\} I_\beta(-v)\frac{dv}{|v|^{1+\alpha}}$$

$$= a(y)f'(y) + (C_\alpha)^\alpha b^\alpha(y)(1-\beta)\int_0^\infty \{f(y-v) - f(y) + f'(y)v\}\frac{dv}{|v|^{1+\alpha}}$$

$$+ \ (C_\alpha)^\alpha b^\alpha(y)(1+\beta)\int_{-\infty}^0 \{f(y-v) - f(y) + f'(y)v\}\frac{dv}{|v|^{1+\alpha}}$$

$$= a(y)f'(y) + (C_\alpha)^\alpha b^\alpha(y)(1-\beta)\left[\frac{\Gamma(2-\alpha)}{\alpha(\alpha-1)}\frac{d^\alpha}{dy^\alpha}f(y)\right]$$

$$+ (C_\alpha)^\alpha b^\alpha(y)(1+\beta)\left[\frac{\Gamma(2-\alpha)}{\alpha(\alpha-1)}\frac{d^\alpha}{d(-y)^\alpha}f(y)\right].$$

Hence, for $1 < \alpha < 2$, the infinitesimal generator of the solution process for SDE in (3.1) can be written as:

$$Af(y) = a(y)f'(y) + \left[(1-\beta)\left(-2\cos\left(\frac{\pi\alpha}{2}\right)\right)^{-1} b^\alpha(y)\frac{d^\alpha}{dy^\alpha}f(y) + (1+\beta)\left(-2\cos\left(\frac{\pi\alpha}{2}\right)\right)^{-1} b^\alpha(y)\frac{d^\alpha}{d(-y)^\alpha}f(y)\right] \quad \text{(E.8)}$$

combine (E.6) and (E.8) to get the infinitesimal generator in (3.2), and the theorem is proved. □

Proof of Theorem 3.2

Case I : $0 < \alpha < 1$ We shall use the forward equation from (2.2b) in this case, i.e,

$$\frac{\partial}{\partial s}p_x(s,y) = \int_{\mathbb{R}_0}\left[(p_x\cdot h)(s,y-r) - (p_x\cdot h)(s,y)\right]\mu(d(-r)) - \frac{\partial}{\partial y}(p_x\cdot H)(s,y)$$

where, $(p_x\cdot h)(s,y) = h(y)p_x(s,y)$; $H(y) = \tilde{a}(y) - K_{\nu_\alpha}b(y)$ and $K_{\nu_\alpha} = \int_{0<|u|<1} u\nu_\alpha(du)$. Note, in this case $H(y) = \tilde{a}(y) - K_{\nu_\alpha}b(y) = a(y)$. Thus, the forward equation:

$$\frac{\partial}{\partial s}p_x(s,y)$$

$$= \int_{\mathbb{R}_0}\left[(p_x\cdot b^\alpha)(s,y-r) - (p_x\cdot b^\alpha)(s,y)\right]\nu_\alpha(d(-r)) - \frac{\partial}{\partial y}(p_x\cdot a)(s,y)$$

$$= (C_\alpha)^\alpha \int_{\mathbb{R}_0}\left[(p_x\cdot b^\alpha)(s,y-r) - (p_x\cdot b^\alpha)(s,y)\right]I_\beta(-r)\frac{dr}{|r|^{1+\alpha}} - \frac{\partial}{\partial y}(p_x\cdot a)(s,y)$$

$$= (1+\beta)(C_\alpha)^\alpha\int_0^\infty\left[(p_x\cdot b^\alpha)(s,y-r) - (p_x\cdot b^\alpha)(s,y)\right]\frac{dr}{|r|^{1+\alpha}}$$

$$+ (1-\beta)(C_\alpha)^\alpha\int_{-\infty}^0\left[(p_x\cdot b^\alpha)(s,y-r) - (p_x\cdot b^\alpha)(s,y)\right]\frac{dr}{|r|^{1+\alpha}} - \frac{\partial}{\partial y}(p_x\cdot a)(s,y)$$

$$= \left[(1+\beta)\left(2\cos\left(\frac{\pi\alpha}{2}\right)\right)^{-1}\frac{d^\alpha}{dy^\alpha}(b^\alpha(y)p_x(s,y))\right.$$

$$\left. + (1-\beta)\left(2\cos\left(\frac{\pi\alpha}{2}\right)\right)^{-1}\frac{d^\alpha}{d(-y)^\alpha}\left(b^\alpha(y)p_x(s,y)\right)\right] - \frac{\partial}{\partial y}\left[a(y)p_x(s,y)\right].$$

Case II : $1 < \alpha < 2$ We shall use the forward equation from (2.2a) in this case, i.e,

$$\frac{\partial}{\partial s} p_x(s, y) = \int_{\mathbb{R}_0} \big[(p_x \cdot h)(y - r, s) - (p_x \cdot h)(s, y) + r(p_x \cdot h)'(y, s)\big]\mu(d(-r))$$
$$- \frac{\partial}{\partial y}(p_x \cdot G)(y, s)$$

where, $(p_x \cdot h)(s, y) = h(y)p_x(s, y)$; $G(y) = \tilde{a}(y) + C_{\nu_\alpha} b(y)$ and $C_{\nu_\alpha} = \int_{|u|\geq 1} u\nu_\alpha(du)$. Note, in this case $G(y) = \tilde{a}(y) + C_{\nu_\alpha} b(y) = a(y)$. Thus, the forward equation:

$$\begin{aligned}
\frac{\partial}{\partial s} p_x(s, y) &= \int_{\mathbb{R}_0} \big[(p_x \cdot b^\alpha)(y - r, s) - (p_x \cdot b^\alpha)(y, s) + r(p_x \cdot b^\alpha)'(y, s)\big]\nu_\alpha(d(-r)) \\
&\quad - \frac{\partial}{\partial y}(p_x \cdot a)(y, s) \\
&= (C_\alpha)^\alpha \int_{\mathbb{R}_0} \big[(p_x \cdot b^\alpha)(s, y - r) - (p_x \cdot b^\alpha)(s, y) + r(p_x \cdot b^\alpha)'(s, y)\big] \\
&\quad \times I_\beta(-r)\frac{dr}{|r|^{1+\alpha}} - \frac{\partial}{\partial y}(p_x \cdot a)(s, y) \\
&= (1 + \beta)(C_\alpha)^\alpha \int_0^\infty \big[(p_x \cdot b^\alpha)(s, y - r) - (p_x \cdot b^\alpha)(s, y) + r(p_x \cdot b^\alpha)'(s, y)\big] \\
&\quad \times \frac{dr}{|r|^{1+\alpha}} \\
&\quad + (1 - \beta)(C_\alpha)^\alpha \int_{-\infty}^0 \big[(p_x \cdot b^\alpha)(s, y - r) - (p_x \cdot b^\alpha)(s, y) \\
&\quad + r(p_x \cdot b^\alpha)'(s, y)\big]\frac{dr}{|r|^{1+\alpha}} - \frac{\partial}{\partial y}(p_x \cdot a)(s, y) \\
&= \Bigg[(1 + \beta)\bigg(-2\cos\Big(\frac{\pi\alpha}{2}\Big)\bigg)^{-1} \frac{d^\alpha}{dy^\alpha}\big(b^\alpha(y)p_x(s, y)\big) + (1 - \beta) \\
&\quad \times \bigg(-2\cos\Big(\frac{\pi\alpha}{2}\Big)\bigg)^{-1} \frac{d^\alpha}{d(-y)^\alpha}\big(b^\alpha(y)p_x(s, y)\big)\Bigg] - \frac{\partial}{\partial y}\big[a(y)p_x(s, y)\big].
\end{aligned}$$

We combine the two cases to get the forward equation of the form (3.4). This concludes the proof. □

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